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Numerical Simulation of Mathematical Model of Fractional Order Partial Differential Equation by Asymptotic Homotopy Perturbatin Method

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ABSTRACT

In the proposed manuscript, a model of the fractional-order partial Cauchy reaction-diffusion equation (CRDE) is solved by a tested and a recent technique, Asymptotic Homotopy Perturbation Method. This technique is a powerful tool for the numerical treatment of various mathematical models of order fractional of linear and non-linear order. CRDE of order-fractional is one of these mathematical models is employed across various fields such as physics, ecology, biology and engineering to model spatial effects and dynamic processes involving both diffusion and chemical reactions. The Caputo order-fractional derivative can also be used for this purpose. Further solution of the concerned model can be provided in the form of a series. The computational work is done by MATLAB software.

1. Introduction

Recently, a significant role has been played by fractional calculus in the field of applied science and engineering. Many real world phenomena can be modeled using fractional derivatives and integrals. In this regard, numerous scientists and researchers advanced fractional calculus due to their scientific thinking and a pivotal contribution. According to the significant role of these scientists and researchers, fractional calculus have got my applications in various fields such as signal processing, traffic systems, damping viscoelasticity, electronics, genetic algorithms, biology, robotics, telecommunications, economics, and finance etc. Further extensive assessment, we recommend the mechanisms presented in [1-8], which also contribute to the best achievement in the development of order-fractional derivatives.

A key focus within this field is the study of order-fractional partial differential equations (FOPDEs), which have been broadly, explored using a variety of procedures tailored to both non-linear and linear forms. For example, the study presented by Necdet Bildik et al. [9] explores the

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analysis of modified Bernoulli sub-equations and nonlinear time-fractional Burgers equations. The numerical simulations of fractional-order space diffusion equations have been discussed in [10-11], represent another important contribution to fractional calculus. Exact solutions for non-linear fractional-order models of biological populations were provided by S. Bushnaq et al. [12] using the numerical scheme Optimal Homotopy Asymptotic Method (OHAM). This method was also a realistic to compute solutions for the Burgers-Huxley models by R. Nawaz et al. [13]. Solutions of system of nonlinear FOPDEs were further examined using another mathematical technique HPTM in [14], while solution of the generalized Mittag-Leffler law via exponential decay was investigated in [15]. A variety of applications involving arbitrary order derivatives as well as integrals were discussed in [16], while numerical schemes and stability analyses for two specific classes of FOPDEs were accessible in [17-18]. Solving both approximate and exact solutions for FOPDEs remains a key focus of ongoing research. In 2019, a well-known technique, Asymptotic Homotopy Perturbation Method (AHPM) was first applied to a special type of FOPDEs, Zakharov-Kuznetsov equation, which models ion-acoustic waves [19]. Following this, the solution of order fractional Helmholtz equations using AHPM [20] marked a significant contribution to the development of fractional calculus.

The primary objective of the current study is to extend the AHPM to report another important fractional-order model Cauchy reaction-diffusion equation (CRDE). The fractional-order variety of the CRDE [21-23] is given by

$$\frac{\partial^{\alpha} Z(\xi,y)}{\partial y^{\alpha}} = c\frac{\partial^{2} Z(\xi,y)}{\partial \xi^{2}} + r(\xi,y)Z(\xi,y), \quad (\xi,y) \in \Omega. \tag{1}$$

when $\alpha = 1$, then Eq. (1) becomes classical reaction-diffusion equation (RDE), where $r(\xi,y)Z(\xi,y)$ is the reaction term, $c\frac{\partial^{2} Z(\xi,y)}{\partial \xi^{2}}$ is the diffusion term and c is the diffusion coefficient. Here, $Z(\xi,y)$ represents the concentration and $r(\xi,y)$ is the reaction parameter of Eq. (1).

2. Preliminaries

Definition 2.1 Order fractional integral of the Riemann--Liouville of a function $h \in L([0,1],R)$ is provided as:

$$I_0^{\alpha} h(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} h(s)ds, \text{ where } 0 < \alpha < 1.$$

Definition 2.2 Derivative of Caputo of order fractional for a function $h \in C_{-1}^{n}$ with $n \in N \cup \{0\}$ is follows as:

$$D_x^{\alpha} h(x) = \begin{cases} I^{n-\alpha} h(x) f^{n}, & n-1 < \alpha \leq n, \quad n \in N, \\ \frac{\partial^n}{\partial x^n} h(x), & \alpha = n, \quad n \in N. \end{cases}$$

Definition 2.3 The Mittag-Leffler function of two parameters is provided as:

$E_{\alpha,\beta}(x) = \sum_{n=o}^{\infty} \frac{x^n}{\Gamma(n\alpha+\beta)}$ If $\alpha = \beta = 1$, then $E_{1,1}(x) = e^{x}$, but $E_{1,1}(-x) = e^{-x}$.

3. Idea of AHPM

In the current portion, we provide the procedure and idea of AHPM for the numerical result and simulation of partial differential equation of order fractional is given by,

$$T\big(Z(\xi,y)\big)+f(\xi,y)=0 \tag{2}$$
$$B\left(Z(\xi,y),\frac{\partial Z(\xi,y)}{\partial y}\right)=0 \tag{3}$$

But $T\big(Z(\xi,y)\big)$ be a differential operator, and $T\big(Z(\xi,y)\big)$ is expressed as:
$$\frac{\partial^\alpha Z(\xi,y)}{\partial y^\alpha}+N\big(Z(\xi,y)\big)+f(\xi,y)=0 \tag{4}$$

With initial condition
$$B\left(Z(\xi,y),\frac{\partial Z(\xi,y)}{\partial y}\right)=0 \tag{5}$$
The differential operator $\frac{\partial^\alpha}{\partial y^\alpha}$ is the derivative of Caputo, but N *consist of* linear or non--linear part of Eq. (4) and B represents the boundary operator, the known function is $f(\xi,y)$, but $Z(\xi,y)$ is the unknown result of Eq. (4), and ξ, y denote temporal and special variables respectively.

Define the Homotopy Ψ $(\xi, y, p) : \Omega \times [0, 1] \rightarrow R$ *that* satisfies the equation
$$\frac{\partial^\alpha \Psi(\xi,y)}{\partial y^\alpha}+f(\xi,y)-p\big[N\big(\Psi(\xi,y;p)\big)\big]=0, \tag{6}$$
where the parameter $p\in[0,1]$ is an embedding parameter. At the current stage the deformation Eq. (6) is the alternate form of the deformation equations as:

$$(1-p)\big[L\big(\Psi(\xi,y;p)\big)-L\big(Z_0(\xi,y)\big)\big]=ph\big[T\big(\Psi(\xi,y;p)\big)+f(\xi,y)\big], \tag{7}$$

$$(1-p)\big[L\big(\Psi(\xi,y;p)\big)-L\big(Z_0(\xi,y)\big)+f(\xi,y)\big]+p\big[T\big(\Psi(\xi,y;p)\big)+f(\xi,y)\big]=0, \tag{8}$$
and
$$(1-p)\big[L\big(\Psi(\xi,y;p)\big)+f(\xi,y)\big]=H(p)\big[T\big(\Psi(\xi,y;p)\big)+f(\xi,y)\big], \tag{9}$$

in HAM, HPM, OHAM recommended by He in [24], Laio in [25] and Marinca in [26] respectively. Due to the definition of Homotopy, when $p=0$ and $p=1$ we have
$\Psi(\xi,y;p)=Z_0(\xi,y)$, $\Psi(\xi,y;p)=Z(\xi,y)$.

Obviously, as *p* varies from 0 to 1, the defined Homotopy guarantees that $\Psi(\xi, y; p)$ convergence to the exact solution $Z(\xi, y)$.

Let define $\Psi(\xi, y; p)$ in the form of
$$\Psi(\xi,y;p)=Z_0(\xi,y)+\textstyle\sum_{i=1}^{\infty} Z_i(\xi,y)\,p^i \tag{10}$$

Moreover, $N(\Psi(\xi,y;p))$ is provided as:
$$N(\Psi(\xi,y;p))=B_1N_0+\textstyle\sum_{i=1}^{\infty}\big(\sum_{m=0}^{i} B_{i+1-m}N_m\big)\,p^i,\ \ where\ \ B_1+B_2+\ldots=-1. \tag{11}$$

And the auxiliary functions are $B_i=B_i(\xi,y;c_i)$, *where* i = 1, 2,… but these auxiliary constants will be discussed later. By taking *p* = 0 to 1 in Eq. (6), then it provides
$\frac{\partial^\alpha Z(\xi,y)}{\partial y^\alpha}+f(\xi,y)=0$ and $\frac{\partial^\alpha Z(\xi,y)}{\partial y^\alpha}+N(Z(\xi,y))+f(\xi,y)=0,$

respectively.

The auxiliary functions or constants in Eq. (11) is clearly different from the constants in [24-26]. Thus, the technique discussed in our article differs from the methods used by He, Liao and Marinca in the aforementioned papers [24-26], as well as from the OHPM presented in [27].

Moreover, if substitute Eq. (10) and (11) in Eq. (6) and after equating the like powers of p, then we obtain the iterative problems are

Top of Form

Bottom of Form

$p^0: \frac{\partial^\alpha Z_0(\xi,y)}{\partial y^\alpha} + f(\xi, y) = 0,$

$p^1: \frac{\partial^\alpha Z_1(\xi,y)}{\partial y^\alpha} = B_1 N_0,$

$p^2: \frac{\partial^\alpha Z_2(\xi,y)}{\partial y^\alpha} = B_2 N_0 + B_1 N_1,$

and *k*th order iteration is

$p^k: \frac{\partial^\alpha Z_k(\xi,y)}{\partial y^\alpha} = \sum_{i=0}^{k-1} B_{k-i} N_i,$

by applying fractional-order integral J^α on both sides of each of the aforementioned sequence problems, we obtain the iterative solutions. Moreover, the convergence of Eq. (10) depends upon $B_i(\xi, y; c_i)$. The selection of $B_i(\xi, y; c_i)$ is based solely on the terms present in the non-linear part of Eq. (2). If *p* = 1 in Eq. (10), then its solution converges to exact solution of Eq. (2).

$$\tilde{Z}(\xi, y) = Z_0(\xi, y) + \sum_{k=1}^{\infty} Z_k(\xi, y; c_i). \qquad (12)$$

Assume, when truncate Eq. (12) into m finite terms to achieve the solution of nonlinear problem. Moreover, the constants terms $c_1, c_2, \ldots.$ are determined by solving the Eq. (13).

$$R(\partial\xi_1, \partial y_1) = R(\partial\xi_2, \partial y_2) = \ldots = R(\partial\xi_k, \partial y_k) = 0, \quad \partial\xi_i, \partial y_i \in [0,1]. \qquad (13)$$

When $p = -p,$ *then* clearly HPM is only a case of Eq. (6) and

$N(\Psi(\xi, y; p)) = N_0 + \sum_{i=1}^{\infty} N_i p^i.$

And when $= ph$, then Eq. (6) is a special case of HAM and

$N(\Psi(\xi, y; p)) = N_0 + \sum_{i=1}^{\infty} N_i p^i.$

OHAM is reduces to a special case for

$B_{k-1} = B_{k-2} + h_k(t, c_j) + \sum_{i=1}^{k-2} h_{k-(i+1)}(t, c_j) B_i, h_k(t, c_j) = c_k$

in Eq. (11). Moreover, when expand and equate the like power of *p* in the deformation equation, we obtain the series solution. Regarding the OHAM, the version proposed in 2008 has undergone multiple enhancements over time. The latest improvements, which include an auxiliary function, are discussed in articles [28] and [29]. Additionally, we have further refined the OHAM by incorporating a very recent auxiliary function, as presented in Eq. (11). Currently, we utilize more recent and general form of the auxiliary function as,

$N(\Psi(\xi, y; p)) = B_1 N_0 + \sum_{i=1}^{\infty} \left(\sum_{m=0}^{i} B_{i+1-m} N_m \right) p^i,$

that depends upon the auxiliary functions $B_1, B_2, B_3, \ldots\ldots$ and is useful for controlling and adjusting the convergence of linear and nonlinear part of the problem with simple way.

4. Application of the Planed Method (AHPM)

In this section AHPM using for numerical treatment of the following order fractional mathematical models.

Problem 1. The AHPM technique for a special case of Eq. (1) at positive y,

$$\frac{\partial^\alpha Z(\xi,y)}{\partial y^\alpha} = \frac{\partial^2 Z(\xi,y)}{\partial \xi^2} - Z(\xi,y), \quad \alpha \in 0,1 \tag{14}$$

Condition is
$Z(\xi,0) = e^{-\xi} + \xi,$

Taking,
$N = -\frac{\partial^2 Z(\xi,y)}{\partial \xi^2} + Z(\xi,y),$

the zeroth-3rd order iterative problems are:

$$\frac{\partial^\alpha Z_0(\xi,y)}{\partial y^\alpha} = 0, Z_0(\xi,0) = e^{-\xi} + \xi,$$
$$\frac{\partial^\alpha Z_1(\xi,y)}{\partial y^\alpha} = B_1 N_0, \quad Z_1(\xi,0) = 0,$$
$$\frac{\partial^\alpha Z_2(\xi,y)}{\partial y^\alpha} = B_2 N_0 + B_1 N_1, \; Z_2(\xi,0) = 0,$$
$$\frac{\partial^\alpha Z_3(\xi,y)}{\partial y^\alpha} = B_3 N_0 + B_2 N_1 + B_1 N_2, \; Z_3(\xi,0) = 0,$$

Therefore, we obtain zeroth-3rd order iterative solutions are

$$Z_0(\xi,y) = e^{-\xi} + \xi,$$
$$Z_1(\xi,y) = B_1 \xi \frac{y^\alpha}{\Gamma(\alpha+1)},$$
$$Z_2(\xi,y) = B_2 \xi \frac{y^\alpha}{\Gamma(\alpha+1)} + {B_1}^2 \xi \frac{y^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$Z_3(\xi,y) = B_3 \xi \frac{y^\alpha}{\Gamma(\alpha+1)} + 2B_1 B_2 \xi \frac{y^{2\alpha}}{\Gamma(2\alpha+1)} + {B_1}^3 \xi \frac{y^{3\alpha}}{\Gamma(3\alpha+1)},$$

assuming $B_i = c_i$ for $i = 1,2,3,4$.

Consider

$\tilde{Z} = Z_0 + Z_1 + \ldots..$

And residual function is

$$R = \frac{\partial^\alpha \tilde{Z}(\xi,y)}{\partial y^\alpha} - \frac{\partial^2 \tilde{Z}(\xi,y)}{\partial \xi^2} + \tilde{Z}(\xi,y), \tag{15}$$

the constant values c_1, c_2 and c_3 can be calculated by using Eq. (13).

Therefore approximate solution of problem 1 can be obtained as:
$\tilde{Z} = Z_0 + Z_1 + Z_2 + Z_3.$

The results of problem 1 are showed in the following tables (Table 1 and Table 2) and figures (Figure 1 and Figure 2), which is show the efficiency of AHPM. Because according to Table 1, for all values of α, the solution decreases gradually with increasing y. This indicates a smooth and stable convergence, which is an important role of AHPM results for the proposed technique.

Table 1
Solution of Problem 1 by AHPM for $c_1 = -0.4916, c_2 = -0.5724, c_3 = 0.0787$ and various values of y at $\xi = 0.005$ and taking $= 0.7, 0.8, 0.9$

y	AHPM (α = 0.7)	AHPM (α = 0.8)	AHPM (α = 0.9)
.05	.999395377321	.999554284043	0.999677808142
.09	.999118686867	.999301516322	0.999457315797
.13	.998898298339	.999085561498	0.999256558351
.17	.998714101427	.998895231326	0.999071060486
.21	.998556527790	.998724938192	0.998898477749
.25	.998419977312	.998571286958	0.998737329785
.29	.998300768299	.998431974164	0.998586577998
.33	.998196289897	.998305320712	0.998445442006
.37	.998104588573	.998190038897	0.998313306346
.41	.998024142667	.998085102997	0.998189667804
.45	.997953729190	.997989671546	0.998074103311
.49	.997892340141	.99790303777	0.997966249195
.53	.997839127341	.997824596467	0.997865787156
.57	.997793364617	.997753821005	0.997772434413

And the context of Table 2, (AHPM) exhibits excellent numerical efficiency and accuracy in solving the given problem. As demonstrated in Table 2, the AHPM results are in close agreement with the exact solution for α = 1, maintaining maximum absolute errors on the order of 10^{-6}. This emphasizes the method's remarkable accuracy, consistency, and stability throughout the whole domain.

Table 2
Absolute errors between exact solution and AHPM results of problem.1 for $c_1 = -0.4916, c_2 = -0.5724, c_3 = 0.0787$ and various values of y at $\xi = 0.005$ and taking $\alpha = 1$

y	Exact (α = 1)	AHPM (α = 1)	Error
0.05	0.999768626315	0.999771169656	$2.543340949840 \times 10^{-6}$
0.09	0.999582135119	0.999585312217	$3.17709781613 \times 10^{-6}$
0.13	0.999402956347	0.999405804872	$2.848524430273 \times 10^{-6}$
0.17	0.999230803276	0.999232609603	$1.80632737478 \times 10^{-6}$
0.21	0.999065400423	0.999065688393	$2.879706130621 \times 10^{-7}$
0.25	0.998906483108	0.998905003224	$1.479883681187 \times 10^{-6}$
0.29	0.998753797031	0.998750516079	$3.280951579740 \times 10^{-6}$
0.33	0.998607097860	0.998602188939	$4.908920463023 \times 10^{-6}$
0.37	0.998466150846	0.998459983788	$6.167058039506 \times 10^{-6}$
0.41	0.998330730443	0.998323862607	$6.867836695665 \times 10^{-6}$
0.45	0.998200619951	0.998193787378	$6.83257257540 \times 10^{-6}$
0.49	0.998075611164	0.998069720085	$5.891078811348 \times 10^{-6}$
0.53	0.997955504041	0.997951622709	$3.881332353177 \times 10^{-6}$
0.57	0.997840106386	0.997839457232	$6.491538597545 \times 10^{-7}$

Furthermore, Figure 1 illustrates that increasing the parameter α from 0.7 to 0.9 enhances the convergence rate and accuracy of the solution.

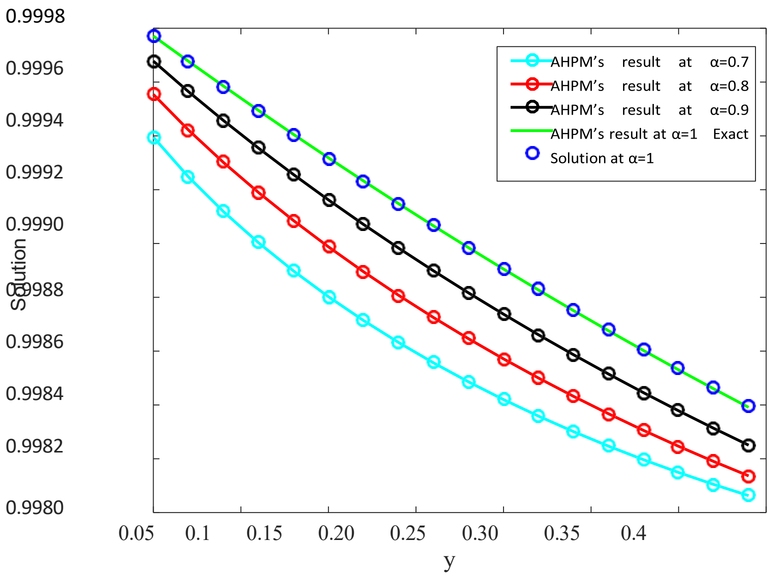

Fig. 1. Exact and AHPM solutions are plotted of Problem 1 at ξ=0.005

Figure 2 shows the absolute errors between AHPM results and exact results, which are closed to each other. This flexibility allows AHPM to be tuned for optimal performance, providing effective control over the convergence behavior without compromising accuracy.

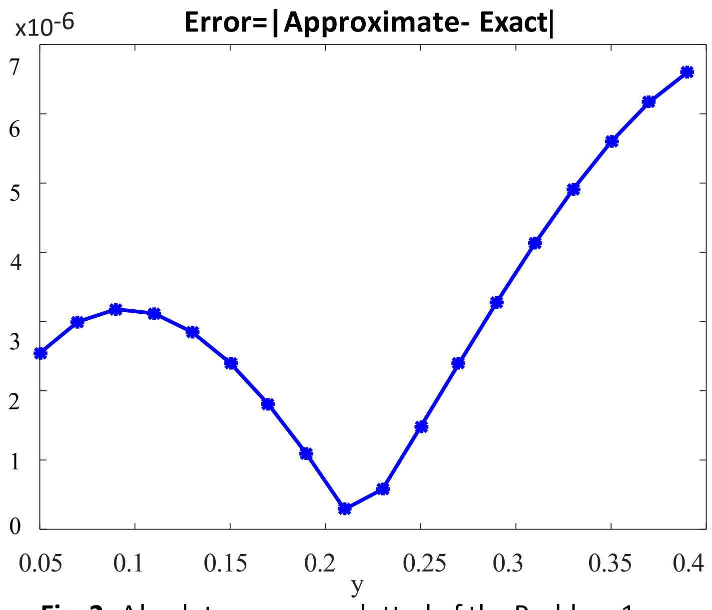

Fig. 2. Absolute error are plotted of the Problem 1 for diverse values of y and $\alpha = 1$.

Problem 2. Consider another special case at y > 0 of CRDE

$$\frac{\partial^\alpha z(\xi,y)}{\partial y^\alpha} = \frac{\partial^2 z(\xi,y)}{\partial \xi^2} - (1 + 4\xi^2)Z(\xi, y), \alpha \in 0,1, \quad (16)$$

initial condition

$Z(\xi,0) = e^{\xi^2}$,
and taking

$$N = -\frac{\partial^2 Z(\xi,y)}{\partial \xi^2} + (1+4\xi^2)Z(\xi,y). \tag{17}$$

Again consider the zeroth-3rd order problems are

$$\frac{\partial^\alpha Z_0(\xi,y)}{\partial y^\alpha} = 0, Z(\xi,0) = e^{\xi^2},$$
$$\frac{\partial^\alpha Z_1(\xi,y)}{\partial y^\alpha} = B_1 N_0, \quad Z_1(\xi,0) = 0,$$
$$\frac{\partial^\alpha Z_2(\xi,y)}{\partial y^\alpha} = B_2 N_0 + B_1 N_1,\ Z_2(\xi,0) = 0,$$
$$\frac{\partial^\alpha Z_3(\xi,y)}{\partial y^\alpha} = B_3 N_0 + B_2 N_1 + B_1 N_2,\ \ Z_3(\xi,0) = 0,$$

Solving these problems by the same method mentioned in problem 1 we get Z_0, Z_1, Z_2 and Z_3, then the series solution is
$\tilde{Z} = Z_0 + Z_1 + Z_2 + Z_3$.

The values of c_1, c_2 and c_3 can be obtained by using Eq. (13).

The solution of problem 2 by using AHPM method is presented in tables (Table 3 and Table 4) and figures (Figure 3 and Figure 4), clearly demonstrate the efficiency of the mentioned technique. For distinct values of α (as $\alpha = 0.7, 0.8, 0.9$) table 3 provides the AHPM results.

Table 3
Results of Problem 2 by AHPM for $c_1 = -0.4916, c_2 = -0.5889, c_3 = 0.0790$ and diverse values of *y* at ξ=0.005 and taking α=0.7, 0.8, 0.9

y	AHPM (α = 0.7)	AHPM (α = 0.8)	AHPM (α = 0.9)
0.005	1.027434139512	1.01565808199	1.008904671063
0.007	1.03484743244	1.020535370022	1.012062137467
0.009	1.041691839703	1.025158676592	1.01513784652
0.011	1.048133616847	1.029598260975	1.01815345217
0.013	1.054269557128	1.033896019741	1.021122187296
0.015	1.060162261679	1.038079568516	1.024052946228
0.017	1.065855153153	1.042168366201	1.026952099101
0.019	1.071379901597	1.046176796843	1.02982442102
0.021	1.076760496404	1.050115884319	1.032673616717
0.023	1.082015655037	1.053994319492	1.03550263891
0.025	1.087160333503	1.057819111518	1.038313892469
0.027	1.09220671852	1.06159601994	1.041109371258
0.029	1.097164904363	1.065329852128	1.043890753211
0.031	1.102043369354	1.069024674407	1.046659468389
0.033	1.106849320366	1.072683965862	1.049416748956
0.035	1.111588947635	1.076310732903	1.052163666672
0.037	1.116267616996	1.079907596268	1.054901161563
0.039	1.12089001745	1.083476858213	1.0576300642

Figure 3 shows the solution gradually converges to the exact solution for $\alpha = 0.7, 0.8, 0.9, 1$.

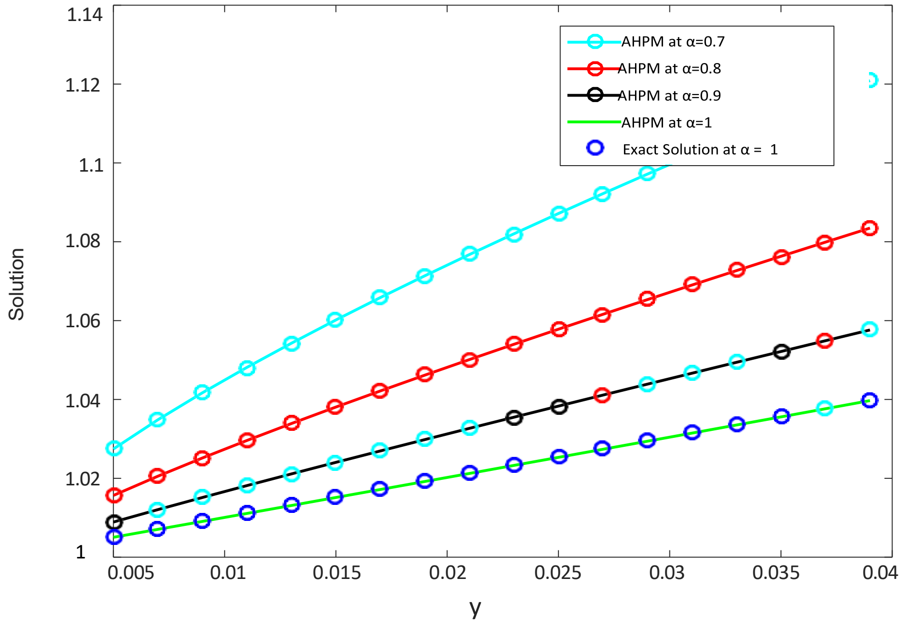

Fig. 3. Exact and AHPM results are plotted of Problem 2 at $\xi = 0.005$ using various values of y and α

Moreover Table 4 and Figure 4 presented the errors of AHPM results and exact results for various values of y at $\alpha = 1$, these results justified that the exact solution and approximate solution by AHPM are closed to each other.

Table 4
Absolute errors can be obtained of exact results and AHPM results of Problem 2 at $\xi = 0.005$ and $\alpha = 1$.

y	Exact (α = 1)	AHPM (α = 1)	Error
0.005	1.005037646486	1.005042886696	$5.240209713163 \times 10^{-6}$
0.007	1.007049733195	1.007055789459	$6.056263737419 \times 10^{-6}$
0.009	1.009065848105	1.009071978339	$6.130234344978 \times 10^{-6}$
0.011	1.011085999279	1.011091454286	$5.455007539256 \times 10^{-6}$
0.013	1.013110194798	1.013114218251	$4.023453178602 \times 10^{-6}$
0.015	1.01513844276	1.015140271185	$1.8284249439799 \times 10^{-6}$
0.017	1.017170751276	1.017169614036	$1.137239693421 \times 10^{-6}$
0.019	1.019207128477	1.019202247758	$4.880719504636 \times 10^{-6}$
0.021	1.021247582508	1.021238173298	$9.409209535446 \times 10^{-6}$
0.023	1.02329212153	1.023277391609	$1.4729921138955 \times 10^{-5}$
0.025	1.025340753723	1.025319903641	$2.085008200824 \times 10^{-5}$
0.027	1.02739348728	1.027365710343	$2.7776936209058 \times 10^{-5}$
0.029	1.029450330412	1.029414812667	$3.551774421264 \times 10^{-5}$
0.031	1.031511291346	1.031467211564	$4.407978292851 \times 10^{-5}$
0.033	1.033576378328	1.033522907982	$5.347034573742 \times 10^{-5}$
0.035	1.035645599616	1.035581902873	$6.369674252431 \times 10^{-5}$
0.037	1.037718963488	1.037644197188	$7.476629971135 \times 10^{-5}$
0.039	1.039796478237	1.039709791877	$8.668636029105 \times 10^{-5}$

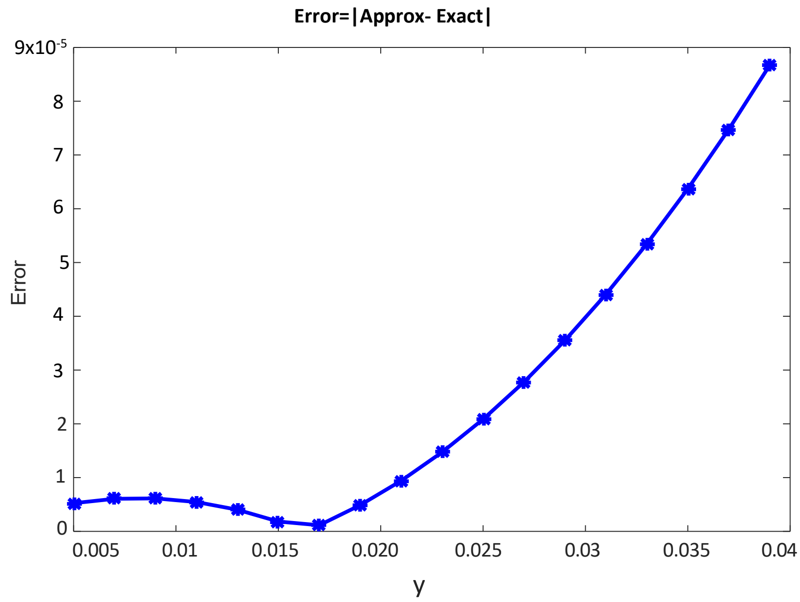

Fig. 4. This figure indicate error estimate of AHPM results of Problem 2 at α=1

Problem 3. The AHPM technique for a special model of FOPDEs, with y > 0

$$\frac{\partial^\alpha Z(\xi,y)}{\partial y^\alpha} = \frac{\partial^2 Z(\xi,y)}{\partial \xi^2} - (2 + 4\xi^2 - 2t)Z(\xi,y),\ \alpha \in 0,1 \tag{18}$$

with initial condition

$Z(\xi,0) = e^{\xi^2}$,

by taking

$L = \frac{\partial^\alpha Z(\xi,y)}{\partial y^\alpha}$,

and

$$N = -\frac{\partial^2 Z(\xi,y)}{\partial \xi^2} + (2 + 4\xi^2 - 2t)Z(\xi,y) \tag{19}$$

Using the above mentioned technique in problem 1 and *taking* $B_1 = c_1, B_2 = c_2$ *and* $B_3 = c_3$, the 3rd order solution can be provided as:

$$\tilde{Z} = Z_0 + Z_1 + Z_2 + Z_3. \tag{20}$$

Residual can be obtained as:

$$R = \frac{\partial^\alpha Z(\xi,y)}{\partial y^\alpha} - \frac{\partial^2 Z(\xi,y)}{\partial \xi^2} + (2 + 4\xi^2 - 2t)Z(\xi,y). \tag{21}$$

For constant values c_1, c_2 and c_3 solving Eq. (13).

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According to the efficiency of AHPM, Table 5 and Figure 5 show AHPM results at distinct values of α and specially Table 6 and Figure 6 indicates the best achievement of AHPM at $\alpha = 1$.

Table 5
The solutions of Problem 3 by AHPM for $c_1 = -0.4836$, $c_2 = -0.5998$, $c_3 = 0.0799$ and different values of y at ξ = 0.05 and taking α = 0.7, 0.8, 0.9

y	AHPM (α = 0.7)	AHPM (α = 0.8)	AHPM (α = 0.9)
0.005	1.002662743991	1.002589703258	1.002549886266
0.007	1.002785955699	1.002661777972	1.002591743839
0.009	1.002936739998	1.002752541022	1.002645983474
0.011	1.003113080369	1.002861069448	1.002712295981
0.013	1.003313485554	1.002986664394	1.00279044184
0.015	1.003536788231	1.003128768312	1.002880226472
0.017	1.003782039113	1.003286920516	1.002981486689
0.019	1.004048444761	1.003460730629	1.003094082464
0.021	1.004335328138	1.003649861513	1.003217891561
0.023	1.004642102134	1.003854017661	1.003352805843
0.025	1.004968250995	1.004072936968	1.003498728624
0.027	1.005313316825	1.004306384687	1.003655572705
0.029	1.005676889499	1.004554148877	1.003823258879
0.031	1.006058598935	1.004816036884	1.004001714771
0.033	1.006458109053	1.005091872569	1.004190873914
0.035	1.006875112978	1.005381494088	1.004390675004
0.037	1.007309329158	1.005684752094	1.004601061292
0.039	1.007760498199	1.006001508247	1.004821980078

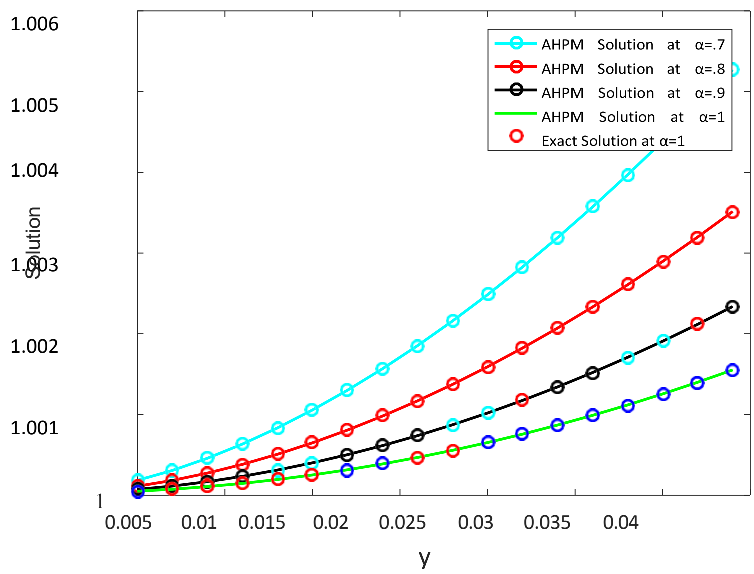

Fig. 5. This figure shows exact and AHPM results of the Problem 3 at ξ = 0.005 for diverse values of y and α

Because in the present of Table 6 and Figure 6 the maximum absolute errors between exact and approximate results around 10^{-6}, which is also show the strong flexibility and excellency of the current technique.

Table 6
Absolute errors of AHPM results of Problem 3 for $c_1 = -0.4836$, $c_2 = -0.5998$, $c_3 = 0.0799$ and distinct values of y at ξ = 0. 05 and taking α = 1

y	Exact solution (α=1)	AHPM solution (α=1)	Error
0.005	1.002528190497	1.002528278158	$8.766074955297 \times 10^{-8}$
0.007	1.002552251463	1.002552423168	$1.717054117282 \times 10^{-7}$
0.009	1.002584333648	1.00258461724	$2.835978429483 \times 10^{-7}$
0.011	1.002624437823	1.002624861018	$4.23194757942 \times 10^{-7}$
0.013	*1*.002672564951	1.002673155268	$5.903169988992 \times 10^{-7}$
0.015	1.002728716187	1.00272950093	$7.847495013211 \times 10^{-7}$
0.017	1.002792892879	1.00279389912	$1.0062412530313 \times 10^{-6}$
0.019	1.002865096566	1.00286635107	$1.254505246353 \times 10^{-6}$
0.021	1.002945328983	1.002946858202	$1.529218423441 \times 10^{-6}$
0.023	1.003129887903	1.003035422077	$1.8300216147798 \times 10^{-6}$
0.025	1.003129887903	1.003132044422	$2.156519470815 \times 10^{-6}$
0.027	1.003234218836	1.003236727117	$2.508280386754 \times 10^{-6}$
0.029	1.003346587361	1.003349472198	$2.884836420492 \times 10^{-6}$
0.031	1.003466996176	1.003470281859	$3.285683203690 \times 10^{-6}$
0.033	1.003595448172	1.003599158452	$3.71027984597 \times 10^{-6}$
0.035	1.003731946435	1.003736104484	$4.158048832254 \times 10^{-6}$
0.037	1.003876494242	1.003881122618	$4.6283759131997 \times 10^{-6}$
0.039	1.004029095067	1.004034215677	$5.120609988774 \times 10^{-6}$

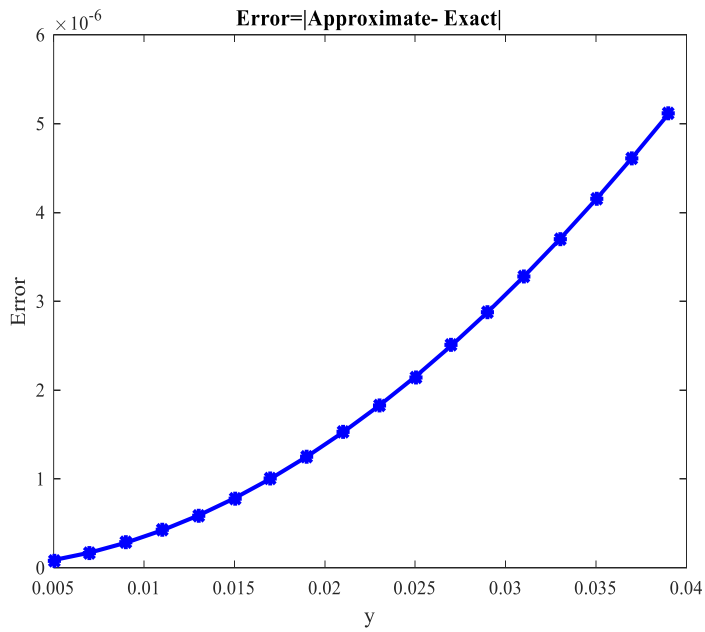

Fig. 6. The absolute error are plotted of Problem 3 at distinct values of y and $\alpha = 1$

5. Discussion

All these discussions and simulations highlight AHPM as an efficient and accurate method for solving fractional-order partial differential equations, which are vital in modelling complex systems in science and engineering. The mathematical technique AHPM acquires a numerical result which is converging and takes minor computational effort than other numerical and analytical techniques.

6. Conclusion

In this article, fractional-order CRDEs have solved by a recent and applicable method OHAM. In the field of engineering and science, the Cauchy reaction-diffusion equation is of extreme importance due to its extensive range of applications. Further, the aforesaid equation's combining diffusion and reaction, with applications in ecology (population dynamics, pattern formation), biology (tumor growth, disease spread), chemistry (reaction kinetics), and material science. Moreover, AHPM is effective and takes minimum time for solving mathematical models. The procedure of AHPM is straightforward, highly accurate and efficient compared to other numerical methods. The simulation and other computational work of this paper have been done using MATLAB software.

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Conflicts of Interest
The authors declare no conflicts of interest.

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